I am calculating the Virasoro algebra explicitly, using only basic commutation relations.

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Definitions

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} : \alpha_n \cdot \alpha_{m-n} :$$

The dot means contracting the Lorentz indices. The modes α_n satisfy the following commutation relation

$$[\alpha^\mu_m, \alpha^\nu_n] = m\,\eta^{\mu\nu}\delta_{m+n,0}.$$

Lemma

$$[L_m, \alpha^\mu_n] = -n\,\alpha^\mu_{n+m}$$

Proof of Lemma

$$\begin{aligned}
[L_m, \alpha^\mu_n] &= \frac{1}{2}\sum_{p=-\infty}^{\infty} [: \alpha_p \cdot \alpha_{m-p} :, \alpha^\mu_n] \\
&= \frac{1}{2}\sum_{p=-\infty}^{m} [\alpha_p \cdot \alpha_{m-p}, \alpha^\mu_n] + \frac{1}{2}\sum_{p=m+1}^{\infty} [\alpha_{m-p} \cdot \alpha_p, \alpha^\mu_n] \\
&= \frac{1}{2}\sum_{p=-\infty}^{m} \left(\alpha_p\,[\alpha_{m-p}, \alpha_n] + [\alpha_p, \alpha_n]\,\alpha_{m-p}\right) + \frac{1}{2}\sum_{p=m+1}^{\infty} \left(\alpha_{m-p}\,[\alpha_p, \alpha_n] + [\alpha_{m-p}, \alpha_n]\,\alpha_p\right) \\
&= \frac{1}{2}\sum_{p=-\infty}^{m} \left(\alpha^\mu_p (m-p)\delta_{m-p+n,0} + p\delta_{p+n,0}\alpha^\mu_{m-p}\right) + \frac{1}{2}\sum_{p=m+1}^{\infty} \left(\alpha^\mu_{m-p} p\delta_{p+n,0} + (m-p)\delta_{m-p+n}\alpha^\mu_p\right) \\
&= \frac{1}{2}\sum_{p=-\infty}^{\infty} \left(\alpha^\mu_p (m-p)\delta_{m-p+n,0} + p\delta_{p+n,0}\alpha^\mu_{m-p}\right) \\
&= \frac{1}{2}\left(\alpha^\mu_{m+n}(-n) + (-n)\alpha^\mu_{m+n}\right) \\
&= -n\alpha^\mu_{m+n}
\end{aligned}$$

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Theorem

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m+n,0}$$

Here D is the space time dimension.

Proof of Theorem I assume wlog $m > 0$. There are two Lorentz indices floating around at maximum. They are contracted if suppressed.

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\begin{aligned}
[L_m, L_n] &= \left[\frac{1}{2}\sum_{q=-\infty}^{\infty} :\alpha_q\cdot\alpha_{m-q}:, L_n\right] \\
&= \left[\frac{1}{2}\sum_{q=-\infty}^{0} :\alpha_q\cdot\alpha_{m-q}:, L_n\right] + \left[\frac{1}{2}\sum_{q=1}^{\infty} :\alpha_q\cdot\alpha_{m-q}:, L_n\right] \\
&= \left[\frac{1}{2}\sum_{q=-\infty}^{0} \alpha_q\cdot\alpha_{m-q}, L_n\right] + \left[\frac{1}{2}\sum_{q=1}^{m} :\alpha_q\cdot\alpha_{m-q}:, L_n\right] + \left[\frac{1}{2}\sum_{q=m+1}^{\infty} :\alpha_q\cdot\alpha_{m-q}:, L_n\right] \\
&= \left[\frac{1}{2}\sum_{q=-\infty}^{0} \alpha_q\cdot\alpha_{m-q}, L_n\right] + \left[\frac{1}{2}\sum_{q=1}^{m} \alpha_q\cdot\alpha_{m-q}, L_n\right] + \left[\frac{1}{2}\sum_{q=m+1}^{\infty} \alpha_{m-q}\cdot\alpha_q, L_n\right] \\
&= \frac{1}{2}\sum_{q=-\infty}^{0}\left(\alpha_q\left[\alpha_{m-q}, L_n\right] + \left[\alpha_q, L_n\right]\alpha_{m-q}\right) \\
&\quad +\frac{1}{2}\sum_{q=1}^{m}\left(\alpha_q\left[\alpha_{m-q}, L_n\right] + \left[\alpha_q, L_n\right]\alpha_{m-q}\right) \\
&\quad +\frac{1}{2}\sum_{q=m+1}^{\infty}\left(\alpha_{m-q}\left[\alpha_q, L_n\right] + \left[\alpha_{m-q}, L_n\right]\alpha_q\right) \\
&= \frac{1}{2}\sum_{q=-\infty}^{0}\left((m-q)\,\alpha_q\alpha_{m+n-q} + q\,\alpha_{n+q}\alpha_{m-q}\right) \\
&\quad +\frac{1}{2}\sum_{q=1}^{m}\left((m-q)\,\alpha_q\alpha_{m+n-q} + q\,\alpha_{n+q}\alpha_{m-q}\right) \\
&\quad +\frac{1}{2}\sum_{q=m+1}^{\infty}\left(q\,\alpha_{m-q}\alpha_{n+q} + D(m-q)\,\alpha_{m+n-q}\alpha_q\right) \\
&= \frac{1}{2}\sum_{q=-\infty}^{0}(m-q):\alpha_q\alpha_{m+n-q}: + \frac{1}{2}\sum_{q=-\infty}^{0} q\ :\alpha_{n+q}\alpha_{m-q}: \\
&\quad +\frac{1}{2}\sum_{q=1}^{m}\left((m-q):\alpha_{m+n-q}\alpha_q: + D(m-q)q\delta_{m+n,0} + q:\alpha_{n+q}\alpha_{m-q}:\right) \\
&\quad +\frac{1}{2}\sum_{q=m+1}^{\infty}\left(q:\alpha_{m-q}\alpha_{n+q}: + (m-q):\alpha_{m+n-q}\alpha_q:\right) \\
&= \frac{1}{2}\sum_{q=-\infty}^{0}(m-q):\alpha_q\alpha_{m+n-q}: + \frac{1}{2}\sum_{q=-\infty}^{0} q\ :\alpha_{n+q}\alpha_{m-q}: \\
&\quad +\frac{1}{2}\sum_{q=1}^{m}(m-q):\alpha_q\alpha_{m+n-q}: + \frac{1}{2}\sum_{q=1}^{m} D(m-q)q\delta_{m+n,0} + \frac{1}{2}\sum_{q=1}^{m} q:\alpha_{n+q}\alpha_{m-q}: \\
&\quad +\frac{1}{2}\sum_{q=m+1}^{\infty} q:\alpha_{m-q}\alpha_{n+q}: + \frac{1}{2}\sum_{q=m+1}^{\infty}(m-q):\alpha_q\alpha_{m+n-q}: \\
&= \frac{1}{2}\sum_{q=-\infty}^{\infty}(m-q):\alpha_q\alpha_{m+n-q}: + \frac{1}{2}\sum_{q=-\infty}^{0} q\ :\alpha_{n+q}\alpha_{m-q}:
\end{aligned}
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\begin{aligned}
& +\frac{1}{2}\sum_{q=1}^{m} D(m-q)q\delta_{m+n,0} + \frac{1}{2}\sum_{q=1}^{m} q :\alpha_{n+q}\alpha_{m-q}: \\
& +\frac{1}{2}\sum_{q=m+1}^{\infty} q :\alpha_{n+q}\alpha_{m-q}: \\
= \;& \frac{1}{2}\sum_{q=-\infty}^{\infty} (m-q) :\alpha_q\alpha_{m+n-q}: + \frac{1}{2}\sum_{q=-\infty}^{\infty} q \;:\alpha_{n+q}\alpha_{m-q}: \\
& +\frac{1}{2}\sum_{q=1}^{m} D(m-q)q\delta_{m+n,0} \\
= \;& \frac{1}{2}\sum_{q=-\infty}^{\infty} (m-q) :\alpha_q\alpha_{m+n-q}: + \frac{1}{2}\sum_{q'=-\infty}^{\infty} (q'-n) \;:\alpha_{q'}\alpha_{m+n-q'}: \\
& +\frac{1}{2}\sum_{q=1}^{m} D(m-q)q\delta_{m+n,0} \\
= \;& \frac{1}{2}\sum_{q=-\infty}^{\infty} (m-n) :\alpha_q\alpha_{m+n-q}: + \frac{1}{2}Dm\sum_{q=1}^{m} q\delta_{m+n,0} - \frac{1}{2}D\sum_{q=1}^{m} q^2\delta_{m+n,0} \\
= \;& (m-n)L_{m+n} + \frac{1}{4}Dm^2(m+1)\delta_{m+n,0} - \frac{1}{12}D(2m^3+3n^2+n)\delta_{m+n,0} \\
= \;& (m-n)L_{m+n} + \frac{1}{12}D(m^3-m)\delta_{m+n,0}
\end{aligned}
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