Aufgabe 6

a) Nach den in Aufgabe 4 gezeigten Regeln gilt:

$$\begin{aligned}
\delta_\epsilon(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\epsilon k^2 + ikx\right) dk \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(\left(i\sqrt{\epsilon}k + \frac{1}{2}\frac{1}{\sqrt{\epsilon}}x\right)^2 - \frac{1}{4}\frac{x^2}{\epsilon}\right) dk \\
&= \frac{1}{2\pi}\exp\left(-\frac{1}{4}\frac{x^2}{\epsilon}\right)\int_{-\infty}^{\infty} \exp\left(\left(i\sqrt{\epsilon}k\right)^2\right) dk \\
&= \frac{1}{2\pi}\exp\left(-\frac{1}{4}\frac{x^2}{\epsilon}\right)\int_{-\infty}^{\infty} \exp\left(-\epsilon k^2\right) dk \\
&= \frac{1}{2\pi}\exp\left(-\frac{1}{4}\frac{x^2}{\epsilon}\right)\frac{\sqrt{\pi}}{\sqrt{\epsilon}} \\
&= \frac{1}{2\sqrt{\pi\epsilon}}\exp\left(-\frac{1}{4}\frac{x^2}{\epsilon}\right)
\end{aligned}$$

b) sowie:

$$\begin{aligned}
\int_{-\infty}^{\infty}\delta_\epsilon(x)dx &= \frac{1}{2\sqrt{\pi\epsilon}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{4}\frac{x^2}{\epsilon}\right)dx \\
&= \frac{1}{2\sqrt{\pi\epsilon}}\int_{-\infty}^{\infty}\exp\left(-\left(\frac{1}{2\sqrt{\epsilon}}\right)^2 x^2\right)dx \\
&= \frac{1}{2\sqrt{\pi\epsilon}}2\sqrt{\pi\epsilon} = 1
\end{aligned}$$

c) Mit der Substitution $x' = y/(2\sqrt{\epsilon})$ ergibt sich:

$$\begin{aligned}
\lim_{\epsilon\to 0}\Theta_\epsilon(x) &= \lim_{\epsilon\to 0}\int_{-\infty}^{x} dy\,\delta_\epsilon(x) \\
&= \lim_{\epsilon\to 0}\int_{-\infty}^{x} dy\frac{1}{2\sqrt{\pi\epsilon}}\exp\left(-\frac{y^2}{4\epsilon}\right) \\
&= \lim_{\epsilon\to 0}\int_{-\infty}^{\frac{x}{2\sqrt{\epsilon}}} dx'\frac{1}{\sqrt{\pi}}\exp\left(-x^2\right) \\
&= \int_{-\infty}^{\mathrm{sign}(x)\infty} dx'\frac{1}{\sqrt{\pi}}\exp\left(-x^2\right) \\
&= \begin{cases} 0 & \text{für x<0} \\ 1 & \text{für x>0} \end{cases}
\end{aligned}$$

d)

Aufgabe 7

a)

$$\begin{aligned} K(x_2,t_2,x_1,t_1=0) &= K_f(x_2,t_2,x_1,t_1=0) - K(-x_2,t_2,x_1,t_1=0)_f \\ &= \sqrt{\frac{m}{2\pi i\hbar}\frac{1}{t}} \cdot \left[\exp\left(\frac{i}{\hbar}\frac{m}{2}\frac{(x_2-x_1)^2}{t_2}\right) - \exp\left(\frac{i}{\hbar}\frac{m}{2}\frac{(x_2+x_1)^2}{t_2}\right)\right] \end{aligned}$$

$$\begin{aligned} \psi(r_2,t_2) &= \int dx_1 K(x_2,t_2,x_1,t_1)\psi(r_1,t_1) \\ &= (4\pi\epsilon)^{-\frac{1}{4}} \int dx_1 K(x_2,t_2,x_1,t_1) \exp\left(-\frac{(x_1-a)^2}{8\epsilon}\right) \\ &= (4\pi\epsilon)^{-\frac{1}{4}} \sqrt{\frac{m}{2\pi i\hbar}\frac{1}{t}} \int dx_1 \\ &\quad \cdot \left[\exp\left(\frac{i}{\hbar}\frac{m}{2}\frac{(x_2-x_1)^2}{t_2}\right) - \exp\left(\frac{i}{\hbar}\frac{m}{2}\frac{(x_2+x_1)^2}{t_2}\right)\right] \exp\left(-\frac{(x_1-a)^2}{8\epsilon}\right) \end{aligned}$$