Aufgabe 9

a)

$$\begin{aligned}
p(x) &= \sum_{i=0}^{n} y_i l_i(x) \\
&= -1 \cdot \frac{(x+1)(x-0)(x-1)}{(-2+1)(-2-0)(-2-1)} + 1 \cdot \frac{(x+2)(x+1)(x-1)}{(0+2)(0+1)(0-1)} + 8 \cdot \frac{(x+2)(x+1)(x-0)}{(1+2)(1+1)(1-0)} \\
&= 1 + 3x + 3x^2 + x^3
\end{aligned}$$

b)

$$\begin{aligned}
y[x_0] &= -1 \\
\delta y[x_0, x_1] &= 1 \\
\delta^2 y[x_0, x_1, x_2] &= 0 \\
\delta^3 y[x_0, x_1, x_2, x_3] &= 1 \\
\Rightarrow p(x) &= 1 + 3x + 3x^2 + x^3
\end{aligned}$$

c)

$$\begin{aligned}
\delta^4 y[x_0, ..., x_4] &= 0 \\
p_4(x) &= p(x) \\
\delta^5 y[x_0, ..., x_5] &= -\frac{49}{360} \\
p_5(x) &= 1 + \frac{221}{90}\, x + \frac{121}{72}\, x^3 + 3\, x^2 - \frac{49}{360}\, x^5
\end{aligned}$$

Aufgabe 10

$$\begin{aligned}
y[t] &= f(t) \\
\delta y[t, t+h/2] &= \frac{f(t+h/2) - f(t)}{h/2} \\
\delta^2 y[t, t+h/2, t+h] &= 2\frac{-2\, f(t+h/2) + f(t) + f(t+h)}{h^2}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow p(x) &= f(t) + (x-t)\left(2\frac{f(t+h/2) - f(t)}{h} + (x-t-h/2)2\frac{-2\, f(t+h/2) + f(t) + f(t+h)}{h^2}\right) \\
&= f(t) + (x-t)\frac{4f(t+h/2) - 3f(t) - f(t+h)}{h} + (x-t)^2\frac{-4f(t+h/2) + 2f(t) + 2f(t+h)}{h^2}
\end{aligned}$$

$$\begin{aligned}
\int_t^{t+h} p(x)dx &= \int_0^h f(t) + x\frac{4f(t+h/2) - 3f(t) - f(t+h)}{h} + x^2\frac{-4f(t+h/2) + 2f(t) + 2f(t+h)}{h^2} dx \\
&= \left[xf(t) + \frac{1}{2}x^2\frac{4f(t+h/2) - 3f(t) - f(t+h)}{h} + \frac{1}{3}x^3\frac{-4f(t+h/2) + 2f(t) + 2f(t+h)}{h^2}\right]_0^h \\
&= hf(t) + \frac{1}{2}h\left(4f(t+h/2) - 3f(t) - f(t+h)\right) + \frac{1}{3}h\left(-4f(t+h/2) + 2f(t) + 2f(t+h)\right) \\
&= \frac{h}{6}\left(f(t) + 4f(t+h/2) + f(t+h)\right)
\end{aligned}$$

Aufgabe 11

$$\begin{aligned} p(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... + a_n x^n \\ &= b_0 - x_0 b_1 + x(b_1 - x_0 b_2 + x(b_2 - x_0 b_3 + x(...(b_n)))) \\ &= b_0 + (x - x_0)\underbrace{(b_1 + b_2 x + b_3 x^2 + ... + b_n x^{n-1})}_{=q(x)} \end{aligned}$$

$$\begin{aligned} p'(x) &= (b_0 + (x - x_0) q(x))' \\ &= q(x) + (x - x_0) q'(x) \\ \Rightarrow & \quad p'(x_0) = q(x_0) \end{aligned}$$

Aufgabe 12

Es gilt:

$$\begin{aligned} f(x) &= p_n(x) + (x - x_0) \cdot ... \cdot (x - x_n) \frac{f^{(n+1)}(\xi)}{(n+1)!} \\ &= f(x_0) + (x - x_0) \cdot \delta f[x_0, x_1] + ... + (x - x_0) \cdot ... \cdot (x - x_{n-1}) \delta^n f[x_0, ..., x_n] \\ &\quad + (x - x_0) \cdot ... \cdot (x - x_n) \frac{f^{(n+1)}(\xi)}{(n+1)!} \end{aligned}$$

Sei $x = x_0 + h$. Im Grenzfall $x_i \to x_0$ gilt:

$$\begin{aligned} f(x_0 + h) &= f(x_0) + \underbrace{(x_0 + h - x_0) \cdot \delta f[x_0, x_1]}_{=h \cdot f'(x_0)} + ... + \underbrace{(x_0 + h - x_0) \cdot ... \cdot (x_0 + h - x_0) \delta^n f[x_0, ..., x_n]}_{=h^n \cdot \frac{f^n(x_0)}{n!}} \\ &\quad + \underbrace{(x - x_0) \cdot ... \cdot (x - x_n)}_{=h^{n+1}} \frac{f^{(n+1)}(\xi)}{(n+1)!} \\ &= \sum_{k=0}^{n} h^k \frac{f^k(x_0)}{k!} + h^{n+1} \frac{f^{(n+1)}(\xi)}{(n+1)!} \end{aligned}$$